

Twisted algebras and Rota-Baxter type operators *

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Abstract

We define the concept of weak pseudotwistor for an algebra (A, μ) in a monoidal category \mathcal{C} , as a morphism $T : A \otimes A \rightarrow A \otimes A$ in \mathcal{C} , satisfying some axioms ensuring that $(A, \mu \circ T)$ is also an algebra in \mathcal{C} . This concept generalizes the previous proposal called pseudotwistor and covers a number of examples of twisted algebras that cannot be covered by pseudotwistors, mainly examples provided by Rota-Baxter operators and some of their relatives (such as Leroux's TD-operators and Reynolds operators). By using weak pseudotwistors, we introduce an equivalence relation (called "twist equivalence") for algebras in a given monoidal category.

1 Introduction

The concept of pseudotwistor (with a particular case called twistor) was introduced in [19] as a general device for twisting (or deforming) the multiplication of an algebra in a monoidal category, obtaining thus a new algebra structure on the same object (informally, we call "twisted algebra" an algebra that can be obtained by deforming the multiplication of a given algebra, maybe with the help of some data on the initial algebra). Namely, if A is an algebra with multiplication $\mu : A \otimes A \rightarrow A$ in a monoidal category \mathcal{C} , a pseudotwistor for A is a morphism $T : A \otimes A \rightarrow A \otimes A$ in \mathcal{C} , such that there exist two morphisms $\tilde{T}_1, \tilde{T}_2 : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ in \mathcal{C} , called the companions of T , satisfying some axioms ensuring that $(A, \mu \circ T)$ is also an algebra in \mathcal{C} . There are many classes of examples of such pseudotwistors. The one that was the starting point of [19] is provided by a twisted tensor product of algebras $A \otimes_R B$ (as in [6], [23]), which is a twisting by a twistor of the ordinary tensor product of algebras $A \otimes B$. Another class of examples is provided by braidings: if c is a braiding on a monoidal category \mathcal{C} , then $c_{A,A}^2$ is a pseudotwistor for every algebra A in \mathcal{C} . The so-called Fedosov product provides another class of examples. Finally, if H is a bialgebra and $\sigma : H \otimes H \rightarrow k$ is a normalized

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and convolution invertible left 2-cocycle, one can consider the twisted algebra ${}_{\sigma}H$, which is the associative algebra structure on H with multiplication $a * b = \sigma(a_1, b_1)a_2b_2$, for all $a, b \in H$; it turns out that this multiplication is afforded by the pseudotwistor

$$T : H \otimes H \rightarrow H \otimes H, \quad T(a \otimes b) = \sigma(a_1, b_1)a_2 \otimes b_2, \quad \forall a, b \in H. \quad (1.1)$$

An indication that a more general concept than pseudotwistors might exist is already implicit in two of the examples given above. For a twisted algebra of the type ${}_{\sigma}H$, the multiplication $*$ is associative even if σ is not convolution invertible, but the map T given by (1.1) is no longer a pseudotwistor in this case. Also, if c is only a pre-braiding on a monoidal category \mathcal{C} (i.e. we do not assume the invertibility of the morphisms $c_{-, -}$) and (A, μ) is an algebra in \mathcal{C} , then $(A, \mu \circ c_{A,A}^2)$ is still an algebra in the category but $c_{A,A}^2$ is no longer a pseudotwistor.

However, we were led to a generalization of pseudotwistors by looking at another class of examples of twisted algebras, provided by Rota-Baxter operators. If (A, μ) is an associative algebra over a field k , with notation $\mu(a \otimes b) = ab$, for $a, b \in A$, and $\lambda \in k$ is a fixed element, a linear map $R : A \rightarrow A$ is called a Rota-Baxter operator of weight λ if it satisfies the relation $R(a)R(b) = R(R(a)b + aR(b) + \lambda ab)$, for all $a, b \in A$. It is well-known that the new multiplication $*_{\lambda}$ on A defined by $a *_{\lambda} b = R(a)b + aR(b) + \lambda ab$ is associative. Also, it is by now well-known that Rota-Baxter operators represent a part of the algebraic component of the Connes-Kreimer approach to renormalization (see [7], [9], [15] and references therein). If we define the linear map $T : A \otimes A \rightarrow A \otimes A$, $T(a \otimes b) = R(a) \otimes b + a \otimes R(b) + \lambda a \otimes b$, for all $a, b \in A$, then the associative multiplication $*_{\lambda}$ may be written as $*_{\lambda} = \mu \circ T$, but T is far from being a pseudotwistor.

Motivated by all these examples, we introduce the following concept. Assume that (A, μ) is an algebra in a monoidal category \mathcal{C} , $T : A \otimes A \rightarrow A \otimes A$ and $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ are morphisms in \mathcal{C} such that:

$$\begin{aligned} T \circ (id_A \otimes (\mu \circ T)) &= (id_A \otimes \mu) \circ \mathcal{T}, \\ T \circ ((\mu \circ T) \otimes id_A) &= (\mu \otimes id_A) \circ \mathcal{T}. \end{aligned}$$

Then $(A, \mu \circ T)$ is also an algebra in \mathcal{C} , denoted by A^T ; the morphism T is called a weak pseudotwistor for A and the morphism \mathcal{T} is called the weak companion of T . It turns out that all the above-mentioned examples of deformed associative multiplications are afforded by weak pseudotwistors, and we provide as well some other examples, coming especially from Rota-Baxter type operators (Reynolds operators, Leroux's TD-operators etc). We present also some general properties of weak pseudotwistors.

A new class of weak pseudotwistors, coming from so-called Rota-Baxter systems, may be found in the recent paper [5]. In fact, as noted by the referee, some of the operators presented in Section 3 of our paper (including the TD-operators) are examples of Rota-Baxter systems, which gives an alternative way of proving that they yield weak pseudotwistors.

In the last section we use weak pseudotwistors in order to introduce an equivalence relation for algebras in a monoidal category \mathcal{C} : if A and B are two such algebras, we say that A and B are twist equivalent (and write $A \equiv_t B$) if there exists an invertible weak pseudotwistor T for A , with invertible weak companion \mathcal{T} , such that A^T and B are isomorphic as algebras. For example, if $A \otimes_R B$ is a twisted tensor product of algebras with bijective twisting map R , then $A \otimes_R B \equiv_t A \otimes B$.

Unless otherwise specified, the (co)algebras that will appear in this paper are *not* supposed to be (co)unital; if A is an associative algebra over a field k we usually denote the multiplication of A by $\mu : A \otimes A \rightarrow A$, $\mu(a \otimes b) = ab$, for all $a, b \in A$. For the composition of two morphisms f and g we write either $g \circ f$ or simply gf . For unexplained terminology we refer to [14].

2 Weak pseudotwistors

We recall the concept of pseudotwistor introduced in [19] (the version for nonunital algebras).

Definition 2.1 *Let (\mathcal{C}, \otimes) be a strict monoidal category, A an algebra in \mathcal{C} with multiplication $\mu : A \otimes A \rightarrow A$ and $T : A \otimes A \rightarrow A \otimes A$ a morphism in \mathcal{C} . Assume that there exist two morphisms $\tilde{T}_1, \tilde{T}_2 : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ in \mathcal{C} such that:*

$$\begin{aligned} T \circ (id_A \otimes \mu) &= (id_A \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_A), \\ T \circ (\mu \otimes id_A) &= (\mu \otimes id_A) \circ \tilde{T}_2 \circ (id_A \otimes T), \\ \tilde{T}_1 \circ (T \otimes id_A) \circ (id_A \otimes T) &= \tilde{T}_2 \circ (id_A \otimes T) \circ (T \otimes id_A). \end{aligned}$$

Then $(A, \mu \circ T)$ is also an algebra in \mathcal{C} , denoted by A^T . The morphism T is called a pseudotwistor and the two morphisms \tilde{T}_1, \tilde{T}_2 are called the companions of T .

We recall from [20] the categorical analogue of the concept of R -matrix introduced in [3], [4] (also the version for nonunital algebras).

Proposition 2.2 *Let (\mathcal{C}, \otimes) be a strict monoidal category, A an algebra in \mathcal{C} with multiplication $\mu : A \otimes A \rightarrow A$ and $T : A \otimes A \rightarrow A \otimes A$ a morphism in \mathcal{C} . Assume that there exist two morphisms $\overline{T}_1, \overline{T}_2 : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ in \mathcal{C} such that:*

$$\begin{aligned} T \circ (id_A \otimes \mu) &= (id_A \otimes \mu) \circ (T \otimes id_A) \circ \overline{T}_1, \\ T \circ (\mu \otimes id_A) &= (\mu \otimes id_A) \circ (id_A \otimes T) \circ \overline{T}_2, \\ (T \otimes id_A) \circ \overline{T}_1 \circ (id_A \otimes T) &= (id_A \otimes T) \circ \overline{T}_2 \circ (T \otimes id_A). \end{aligned}$$

Then $(A, \mu \circ T)$ is also an algebra in \mathcal{C} , denoted by A^T . The morphism T is called an R -matrix and the two morphisms $\overline{T}_1, \overline{T}_2$ are called the companions of T .

We introduce now a common generalization of these two concepts.

Theorem 2.3 *Let (\mathcal{C}, \otimes) be a strict monoidal category, A an algebra in \mathcal{C} with multiplication $\mu : A \otimes A \rightarrow A$ and $T : A \otimes A \rightarrow A \otimes A$ a morphism in \mathcal{C} . Assume that there exists a morphism $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ in \mathcal{C} such that:*

$$T \circ (id_A \otimes (\mu \circ T)) = (id_A \otimes \mu) \circ \mathcal{T}, \quad (2.1)$$

$$T \circ ((\mu \circ T) \otimes id_A) = (\mu \otimes id_A) \circ \mathcal{T}. \quad (2.2)$$

Then $(A, \mu \circ T)$ is also an algebra in \mathcal{C} , denoted by A^T . The morphism T is called a weak pseudotwistor and the morphism \mathcal{T} is called the weak companion of T .

Proof. We compute:

$$\begin{aligned} (\mu \circ T) \circ ((\mu \circ T) \otimes id_A) &= \mu \circ T \circ ((\mu \circ T) \otimes id_A) \\ &\stackrel{(2.2)}{=} \mu \circ (\mu \otimes id_A) \circ \mathcal{T} \\ &= \mu \circ (id_A \otimes \mu) \circ \mathcal{T} \\ &\stackrel{(2.1)}{=} (\mu \circ T) \circ (id_A \otimes (\mu \circ T)), \end{aligned}$$

finishing the proof. □

Remark 2.4 If T is a pseudotwistor with companions \tilde{T}_1, \tilde{T}_2 on an algebra A , then T is also a weak pseudotwistor, with weak companion $\mathcal{T} = \tilde{T}_1 \circ (T \otimes id_A) \circ (id_A \otimes T) = \tilde{T}_2 \circ (id_A \otimes T) \circ (T \otimes id_A)$.

Conversely, an invertible weak pseudotwistor T with weak companion \mathcal{T} on an algebra A is a pseudotwistor, with companions $\tilde{T}_1 = \mathcal{T} \circ (id_A \otimes T^{-1}) \circ (T^{-1} \otimes id_A)$ and $\tilde{T}_2 = \mathcal{T} \circ (T^{-1} \otimes id_A) \circ (id_A \otimes T^{-1})$.

If T is an R -matrix with companions \bar{T}_1, \bar{T}_2 on an algebra A , then T is a weak pseudotwistor, with weak companion $\mathcal{T} = (T \otimes id_A) \circ \bar{T}_1 \circ (id_A \otimes T) = (id_A \otimes T) \circ \bar{T}_2 \circ (T \otimes id_A)$.

Conversely, an invertible weak pseudotwistor T with weak companion \mathcal{T} on an algebra A is an R -matrix, with companions $\bar{T}_1 = (T^{-1} \otimes id_A) \circ \mathcal{T} \circ (id_A \otimes T^{-1})$ and $\bar{T}_2 = (id_A \otimes T^{-1}) \circ \mathcal{T} \circ (T^{-1} \otimes id_A)$.

Example 2.5 Let A be an associative unital algebra with unit 1_A over a field k . In [16] the following linear map was considered:

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = 1_A \otimes ab, \quad \forall a, b \in A.$$

The associativity of the multiplication of A is equivalent to the fact that this map T is a Yang-Baxter operator, cf. [16].

One can check, by a direct computation, that T is a pseudotwistor (in particular, a weak pseudotwistor) with companions $\tilde{T}_1 = id_A \otimes T$ and $\tilde{T}_2(a \otimes b \otimes c) = 1_A \otimes 1_A \otimes abc$, for all $a, b, c \in A$, and obviously $A^T = A$.

Remark 2.6 It is possible to have an associative algebra A over a field k , a linear map $T : A \otimes A \rightarrow A \otimes A$ that is not a weak pseudotwistor but such that $\mu \circ T$ is associative. Indeed, for a given associative algebra A , the linear map $T : A \otimes A \rightarrow A \otimes A$, $T(a \otimes b) = b \otimes a$, for all $a, b \in A$, has the property that $\mu \circ T$ is associative (of course, it is just the multiplication of A^{op}) but, in general, T is not a weak pseudotwistor (although it may be, for some particular algebras A). We give a concrete example of an associative algebra A for which T is not a weak pseudotwistor. We take A to be a 2-dimensional associative algebra over a field k with a linear basis $\{x, y\}$ with multiplication table $x \cdot x = x \cdot y = y \cdot x = y \cdot y = x$. We claim that there exists no linear map $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ such that $T \circ (id_A \otimes (\mu \circ T))(y \otimes x \otimes x) = (id_A \otimes \mu) \circ \mathcal{T}(y \otimes x \otimes x)$. Suppose that such a map exists. We have $T \circ (id_A \otimes (\mu \circ T))(y \otimes x \otimes x) = x \otimes y$. If we write $\mathcal{T}(y \otimes x \otimes x) = a_1 x \otimes x \otimes x + a_2 x \otimes x \otimes y + a_3 x \otimes y \otimes x + a_4 x \otimes y \otimes y + a_5 y \otimes x \otimes x + a_6 y \otimes x \otimes y + a_7 y \otimes y \otimes x + a_8 y \otimes y \otimes y$, with $a_1, \dots, a_8 \in k$, then we have $(id_A \otimes \mu) \circ \mathcal{T}(y \otimes x \otimes x) = (a_1 + a_2 + a_3 + a_4)x \otimes x + (a_5 + a_6 + a_7 + a_8)y \otimes x$, and this element cannot be equal to $x \otimes y$.

Let (\mathcal{C}, \otimes) be a strict monoidal category and $T_{X,Y} : X \otimes Y \rightarrow X \otimes Y$ a family of natural morphisms in \mathcal{C} such that, for all $X, Y, Z \in \mathcal{C}$, we have:

$$T_{X \otimes Y, Z} \circ (T_{X,Y} \otimes id_Z) = T_{X, Y \otimes Z} \circ (id_X \otimes T_{Y,Z}). \quad (2.3)$$

It was proved in [20] that if for all $X, Y \in \mathcal{C}$ the morphism $T_{X,Y}$ is an isomorphism, then, for every algebra (A, μ) in \mathcal{C} , the morphism $T_{A,A} : A \otimes A \rightarrow A \otimes A$ is a pseudotwistor. If we do not assume the invertibility of the morphisms $T_{X,Y}$, then $T_{A,A}$ is no longer a pseudotwistor.

Proposition 2.7 $T_{A,A}$ is a weak pseudotwistor.

Proof. The identity (2.3) for $X = Y = Z = A$ becomes $T_{A \otimes A, A} \circ (T_{A,A} \otimes id_A) = T_{A, A \otimes A} \circ (id_A \otimes T_{A,A})$; we denote by \mathcal{T} this morphism from $A \otimes A \otimes A$ to $A \otimes A \otimes A$. The naturality of T implies that $T_{A,A} \circ (id_A \otimes \mu) = (id_A \otimes \mu) \circ T_{A, A \otimes A}$ and $T_{A,A} \circ (\mu \otimes id_A) = (\mu \otimes id_A) \circ T_{A \otimes A, A}$, hence:

$$T_{A,A} \circ (id_A \otimes (\mu \circ T_{A,A})) = (id_A \otimes \mu) \circ T_{A, A \otimes A} \circ (id_A \otimes T_{A,A}) = (id_A \otimes \mu) \circ \mathcal{T},$$

$$T_{A,A} \circ ((\mu \circ T_{A,A}) \otimes id_A) = (\mu \otimes id_A) \circ T_{A \otimes A, A} \circ (T_{A,A} \otimes id_A) = (\mu \otimes id_A) \circ \mathcal{T},$$

proving that $T_{A,A}$ is a weak pseudotwistor with weak companion \mathcal{T} . \square

Definition 2.8 ([6], [23]) Let (\mathcal{C}, \otimes) be a strict monoidal category and (A, μ_A) , (B, μ_B) two algebras in \mathcal{C} . A twisting map between A and B is a morphism $R : B \otimes A \rightarrow A \otimes B$ in \mathcal{C} satisfying the conditions:

$$R \circ (id_B \otimes \mu_A) = (\mu_A \otimes id_B) \circ (id_A \otimes R) \circ (R \otimes id_A), \quad (2.4)$$

$$R \circ (\mu_B \otimes id_A) = (id_A \otimes \mu_B) \circ (R \otimes id_B) \circ (id_B \otimes R). \quad (2.5)$$

The next result generalizes [19], Theorem 6.6 (iii).

Proposition 2.9 Let (\mathcal{C}, \otimes) be a strict monoidal category and (A, μ) an algebra in \mathcal{C} . Assume that $Q, P : A \otimes A \rightarrow A \otimes A$ are two twisting maps between A and itself, such that:

$$(P \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A) = (id_A \otimes P) \circ (P \otimes id_A) \circ (id_A \otimes P), \quad (2.6)$$

$$(Q \otimes id_A) \circ (id_A \otimes Q) \circ (Q \otimes id_A) = (id_A \otimes Q) \circ (Q \otimes id_A) \circ (id_A \otimes Q), \quad (2.7)$$

$$(P \otimes id_A) \circ (id_A \otimes P) \circ (Q \otimes id_A) = (id_A \otimes Q) \circ (P \otimes id_A) \circ (id_A \otimes P), \quad (2.8)$$

$$(Q \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A) = (id_A \otimes P) \circ (P \otimes id_A) \circ (id_A \otimes Q). \quad (2.9)$$

Then $T := Q \circ P : A \otimes A \rightarrow A \otimes A$ is a weak pseudotwistor.

Proof. Because of (2.6) and (2.7), we have the following equality:

$$\begin{aligned} (Q \otimes id_A) \circ (id_A \otimes Q) \circ (Q \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A) \circ (id_A \otimes P) \\ = (id_A \otimes Q) \circ (Q \otimes id_A) \circ (id_A \otimes Q) \circ (P \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A). \end{aligned}$$

This morphism from $A \otimes A \otimes A$ to $A \otimes A \otimes A$ will be denoted by \mathcal{T} . Now we compute:

$$\begin{aligned} T \circ (id_A \otimes (\mu \circ T)) &= Q \circ P \circ (id_A \otimes \mu) \circ (id_A \otimes Q) \circ (id_A \otimes P) \\ &\stackrel{(2.4)}{=} Q \circ (\mu \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A) \circ (id_A \otimes Q) \circ (id_A \otimes P) \\ &\stackrel{(2.9)}{=} Q \circ (\mu \otimes id_A) \circ (Q \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A) \circ (id_A \otimes P) \\ &\stackrel{(2.5)}{=} (id_A \otimes \mu) \circ (Q \otimes id_A) \circ (id_A \otimes Q) \circ (Q \otimes id_A) \circ (id_A \otimes P) \\ &\quad \circ (P \otimes id_A) \circ (id_A \otimes P) \\ &= (id_A \otimes \mu) \circ \mathcal{T}, \end{aligned}$$

$$\begin{aligned} T \circ ((\mu \circ T) \otimes id_A) &= Q \circ P \circ (\mu \otimes id_A) \circ (Q \otimes id_A) \circ (P \otimes id_A) \\ &\stackrel{(2.5)}{=} Q \circ (id_A \otimes \mu) \circ (P \otimes id_A) \circ (id_A \otimes P) \circ (Q \otimes id_A) \circ (P \otimes id_A) \\ &\stackrel{(2.8)}{=} Q \circ (id_A \otimes \mu) \circ (id_A \otimes Q) \circ (P \otimes id_A) \circ (id_A \otimes P) \circ (P \otimes id_A) \\ &\stackrel{(2.4)}{=} (\mu \otimes id_A) \circ (id_A \otimes Q) \circ (Q \otimes id_A) \circ (id_A \otimes Q) \circ (P \otimes id_A) \\ &\quad \circ (id_A \otimes P) \circ (P \otimes id_A) \\ &= (\mu \otimes id_A) \circ \mathcal{T}, \end{aligned}$$

so T is a weak pseudotwistor with weak companion \mathcal{T} . \square

Corollary 2.10 *Let (\mathcal{C}, \otimes) be a strict monoidal category and $c_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ a pre-braiding on \mathcal{C} (that is c satisfies all the axioms of a braiding as in [14] except for the fact that we do not require $c_{X,Y}$ to be invertible). Let (A, μ) be an algebra in \mathcal{C} . Then $c_{A,A}^2 = c_{A,A} \circ c_{A,A} : A \otimes A \rightarrow A \otimes A$ is a weak pseudotwistor.*

Proof. It is either a consequence of Proposition 2.7, by noting that the family of natural morphisms $T_{X,Y} := c_{Y,X} \circ c_{X,Y}$ satisfies (2.3), or a consequence of Proposition 2.9, applied to the twisting maps $Q = P = c_{A,A}$. \square

Example 2.11 *Let H be a bialgebra over a field k , with multiplication $\mu : H \otimes H \rightarrow H$, $\mu(h \otimes h') = hh'$, for $h, h' \in H$, and comultiplication $\Delta : H \rightarrow H \otimes H$, for which we use a Sweedler-type notation $\Delta(h) = h_1 \otimes h_2$, for $h \in H$. Let $\sigma : H \otimes H \rightarrow k$ be a left 2-cocycle, that is we have $\sigma(a_1, b_1)\sigma(a_2 b_2, c) = \sigma(b_1, c_1)\sigma(a, b_2 c_2)$, for all $a, b, c \in H$. It is well-known that, if we define a new multiplication on H , by $a * b = \sigma(a_1, b_1)a_2 b_2$, for all $a, b \in H$, then this multiplication is associative and the new algebra structure on H is denoted by ${}_{\sigma}H$.*

Define the linear map $T : H \otimes H \rightarrow H \otimes H$, $T(a \otimes b) = \sigma(a_1, b_1)a_2 \otimes b_2$, for all $a, b \in H$. It was proved in [19] that, if H is unital and counital and σ is convolution invertible with inverse σ^{-1} , then T is a pseudotwistor, with companions $\tilde{T}_1, \tilde{T}_2 : H \otimes H \otimes H \rightarrow H \otimes H \otimes H$, $\tilde{T}_1(a \otimes b \otimes c) = \sigma^{-1}(a_1, b_1)\sigma(a_2, b_2 c_1)a_3 \otimes b_3 \otimes c_2$ and $\tilde{T}_2(a \otimes b \otimes c) = \sigma^{-1}(b_1, c_1)\sigma(a_1 b_2, c_2)a_2 \otimes b_3 \otimes c_3$.

If σ is not convolution invertible, T is no longer a pseudotwistor. However, T is a weak pseudotwistor, with weak companion $\mathcal{T} : H \otimes H \otimes H \rightarrow H \otimes H \otimes H$, $\mathcal{T}(a \otimes b \otimes c) = \sigma(b_1, c_1)\sigma(a_1, b_2 c_2)a_2 \otimes b_3 \otimes c_3 = \sigma(a_1, b_1)\sigma(a_2 b_2, c_1)a_3 \otimes b_3 \otimes c_2$, for all $a, b, c \in H$, as one can easily check.

*There exist "mirror versions" of these facts. Namely, let $\tau : H \otimes H \rightarrow k$ be a right 2-cocycle, i.e. τ satisfies the condition $\tau(a_1 b_1, c)\tau(a_2, b_2) = \tau(a, b_1 c_1)\tau(b_2, c_2)$, for all $a, b, c \in H$. If we define a new multiplication on H by $a * b = a_1 b_1 \tau(a_2, b_2)$, for all $a, b \in H$, then this multiplication is associative and the new algebra structure is denoted by H_{τ} . This multiplication is afforded by the weak pseudotwistor $D : H \otimes H \rightarrow H \otimes H$, $D(a \otimes b) = a_1 \otimes b_1 \tau(a_2, b_2)$, whose weak companion is the linear map $\mathcal{D} : H \otimes H \otimes H \rightarrow H \otimes H \otimes H$, $\mathcal{D}(a \otimes b \otimes c) = a_1 \otimes b_1 \otimes c_1 \tau(a_2, b_2 c_2)\tau(b_3, c_3) = a_1 \otimes b_1 \otimes c_1 \tau(a_2 b_2, c_2)\tau(a_3, b_3)$.*

Example 2.12 *Let A be an associative algebra with unit 1 over a field k and $\lambda, \theta, \nu \in k$ some fixed scalars. In [8] the following linear map was considered:*

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = \lambda ab \otimes 1 + \theta 1 \otimes ab - \nu a \otimes b, \quad \forall a, b \in A.$$

Then one can easily check that T is a weak pseudotwistor with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, $\mathcal{T}(a \otimes b \otimes c) = (\lambda + \theta - \nu)(\lambda abc \otimes 1 \otimes 1 + \theta 1 \otimes 1 \otimes abc - \nu a \otimes b \otimes c)$, for all $a, b, c \in A$.

Proposition 2.13 *Let (A, μ) be an associative algebra. We regard $A \otimes A$ as an A -bimodule in the usual way: $a \cdot (b \otimes c) \cdot d = ab \otimes cd$, for all $a, b, c, d \in A$. Let $\delta : A \rightarrow A \otimes A$ be a linear map, with Sweedler-type notation $\delta(a) = a_1 \otimes a_2$, that is a morphism of A -bimodules, i.e.*

$$\delta(ab) = ab_1 \otimes b_2, \tag{2.10}$$

$$\delta(ab) = a_1 \otimes a_2 b, \tag{2.11}$$

for all $a, b \in A$. Define the linear map $T : A \otimes A \rightarrow A \otimes A$, $T(a \otimes b) = b_1 a \otimes b_2$. Then T satisfies the following relations (with standard notation for T_{12}, T_{13}, T_{23}):

$$T \circ (id_A \otimes \mu) = (id_A \otimes \mu) \circ T_{12}, \tag{2.12}$$

$$T \circ (\mu \otimes id_A) = (\mu \otimes id_A) \circ T_{13}, \quad (2.13)$$

$$T_{12} \circ T_{23} = T_{13} \circ T_{12}. \quad (2.14)$$

Consequently, T is a weak pseudotwistor with weak companion $\mathcal{T} = T_{12} \circ T_{23} = T_{13} \circ T_{12}$, and the new multiplication on A defined by $a * b = b_1 a b_2$ is associative.

Proof. We compute:

$$\begin{aligned} T(a \otimes bc) &= (bc)_1 a \otimes (bc)_2 \\ &\stackrel{(2.11)}{=} b_1 a \otimes b_2 c = ((id_A \otimes \mu) \circ T_{12})(a \otimes b \otimes c), \end{aligned}$$

$$T(ab \otimes c) = c_1 ab \otimes c_2 = ((\mu \otimes id_A) \circ T_{13})(a \otimes b \otimes c),$$

$$\begin{aligned} (T_{12} \circ T_{23})(a \otimes b \otimes c) &= T_{12}(a \otimes c_1 b \otimes c_2) \\ &= (c_1 b)_1 a \otimes (c_1 b)_2 \otimes c_2 \\ &\stackrel{(2.10)}{=} c_1 b_1 a \otimes b_2 \otimes c_2 \\ &= T_{13}(b_1 a \otimes b_2 \otimes c) \\ &= (T_{13} \circ T_{12})(a \otimes b \otimes c), \end{aligned}$$

finishing the proof. \square

Remark 2.14 The definition of the multiplication $*$ in Proposition 2.13 is inspired by the result of Aguiar from [1], showing that if (A, μ, Δ) is an infinitesimal bialgebra (i.e. $\Delta : A \rightarrow A \otimes A$ is a coassociative derivation) then the new product on A defined by $a \circ b = b_1 a b_2$ is pre-Lie.

Remark 2.15 The referee suggested the following extension of Proposition 2.13. Consider the data (A, ν, σ, δ) , where A is an associative algebra, $\nu, \sigma : A \rightarrow A$ are algebra automorphisms and $\delta : A \rightarrow A \otimes A$ is an A -bimodule map, where the A -bimodule structure of $A \otimes A$ is now twisted by ν and σ , that is

$$a \cdot (b \otimes c) \cdot d = \nu(a) b \otimes c \sigma(d), \quad \forall a, b, c, d \in A.$$

By using the same Sweedler-type notation for δ , namely $\delta(a) = a_1 \otimes a_2$, define the linear map $T : A \otimes A \rightarrow A \otimes A$, $T(a \otimes b) = \nu^{-1}(b_1) a \otimes \sigma^{-1}(b_2)$. Then T satisfies the relations (2.12)-(2.14), hence it is also a weak pseudotwistor with weak companion $\mathcal{T} = T_{12} \circ T_{23} = T_{13} \circ T_{12}$, and the new multiplication defined on A by $a * b = \nu^{-1}(b_1) a \sigma^{-1}(b_2)$ is associative.

Proposition 2.16 Let (\mathcal{C}, \otimes) be a strict monoidal category and (A, μ) an algebra in \mathcal{C} . Assume that T and D are two weak pseudotwistors for A , with weak companions \mathcal{T} and respectively \mathcal{D} , such that the following conditions are satisfied:

$$D \circ (id_A \otimes (\mu \circ T \circ D)) = (id_A \otimes (\mu \circ T)) \circ \mathcal{D}, \quad (2.15)$$

$$D \circ ((\mu \circ T \circ D) \otimes id_A) = ((\mu \circ T) \otimes id_A) \circ \mathcal{D}. \quad (2.16)$$

Then $T \circ D$ is a weak pseudotwistor for A , with weak companion $\mathcal{T} \circ \mathcal{D}$.

Proof. We compute:

$$\begin{aligned}
T \circ D \circ (id_A \otimes (\mu \circ T \circ D)) &\stackrel{(2.15)}{=} T \circ (id_A \otimes (\mu \circ T)) \circ \mathcal{D} \\
&\stackrel{(2.1)}{=} (id_A \otimes \mu) \circ \mathcal{T} \circ \mathcal{D}, \\
\\
T \circ D \circ ((\mu \circ T \circ D) \otimes id_A) &\stackrel{(2.16)}{=} T \circ ((\mu \circ T) \otimes id_A) \circ \mathcal{D} \\
&\stackrel{(2.2)}{=} (\mu \otimes id_A) \circ \mathcal{T} \circ \mathcal{D},
\end{aligned}$$

finishing the proof. \square

Corollary 2.17 *Let (\mathcal{C}, \otimes) be a strict monoidal category and (A, μ) an algebra in \mathcal{C} . Assume that T and D are two weak pseudotwistors for A , with weak companions \mathcal{T} and respectively \mathcal{D} , such that the following conditions are satisfied:*

$$\mu \circ T \circ D = \mu \circ D \circ T, \quad (2.17)$$

$$\mathcal{D} \circ (id_A \otimes T) = (id_A \otimes T) \circ \mathcal{D}, \quad (2.18)$$

$$\mathcal{D} \circ (T \otimes id_A) = (T \otimes id_A) \circ \mathcal{D}. \quad (2.19)$$

Then $T \circ D$ is a weak pseudotwistor for A , with weak companion $\mathcal{T} \circ \mathcal{D}$.

Proof. We check (2.15), while (2.16) is similar and left to the reader:

$$\begin{aligned}
D \circ (id_A \otimes (\mu \circ T \circ D)) &\stackrel{(2.17)}{=} D \circ (id_A \otimes (\mu \circ D \circ T)) \\
&= D \circ (id_A \otimes (\mu \circ D)) \circ (id_A \otimes T) \\
&\stackrel{(2.1)}{=} (id_A \otimes \mu) \circ \mathcal{D} \circ (id_A \otimes T) \\
&\stackrel{(2.18)}{=} (id_A \otimes \mu) \circ (id_A \otimes T) \circ \mathcal{D} \\
&= (id_A \otimes (\mu \circ T)) \circ \mathcal{D},
\end{aligned}$$

finishing the proof. \square

Let H be a bialgebra over a field k as in Example 2.11, σ (respectively τ) a left (respectively right) 2-cocycle on H and T and D the weak pseudotwistors defined in Example 2.11. The multiplication defined on H by

$$a * b = \sigma(a_1, b_1) a_2 b_2 \tau(a_3, b_3), \quad \forall a, b \in H, \quad (2.20)$$

is associative. We can obtain this as consequence of Corollary 2.17. It is obvious that $T \circ D = D \circ T$, so (2.17) is satisfied. It is easy to see that (2.18) and (2.19) are satisfied too, so $T \circ D$ is a weak pseudotwistor and clearly $\mu \circ T \circ D$ is exactly the multiplication $*$ defined by (2.20).

We present now another application of Proposition 2.16, generalizing Remark 4.11 in [20]:

Proposition 2.18 *Let (\mathcal{C}, \otimes) be a strict monoidal category, (A, μ) an algebra in \mathcal{C} , $T : A \otimes A \rightarrow A \otimes A$ a weak pseudotwistor with weak companion \mathcal{T} and $D_{X,Y} : X \otimes Y \rightarrow X \otimes Y$ a family of natural morphisms in \mathcal{C} satisfying (2.3). Then $T \circ D_{A,A}$ is a weak pseudotwistor for A .*

Proof. The naturality of $D_{X,Y}$ implies:

$$\begin{aligned} D_{A,A} \circ (id_A \otimes (\mu \circ T)) &= (id_A \otimes (\mu \circ T)) \circ D_{A,A \otimes A}, \\ D_{A,A} \circ ((\mu \circ T) \otimes id_A) &= ((\mu \circ T) \otimes id_A) \circ D_{A \otimes A, A}. \end{aligned}$$

By composing on the right with $id_A \otimes D_{A,A}$ and respectively $D_{A,A} \otimes id_A$ we obtain:

$$\begin{aligned} D_{A,A} \circ (id_A \otimes (\mu \circ T \circ D_{A,A})) &= (id_A \otimes (\mu \circ T)) \circ D_{A,A \otimes A} \circ (id_A \otimes D_{A,A}), \\ D_{A,A} \circ ((\mu \circ T \circ D_{A,A}) \otimes id_A) &= ((\mu \circ T) \otimes id_A) \circ D_{A \otimes A, A} \circ (D_{A,A} \otimes id_A). \end{aligned}$$

The weak companion of $D_{A,A}$ is $\mathcal{D} = D_{A,A \otimes A} \circ (id_A \otimes D_{A,A}) = D_{A \otimes A, A} \circ (D_{A,A} \otimes id_A)$, so we obtained (2.15) and (2.16). \square

Let (\mathcal{C}, \otimes) be a strict monoidal category, (A, μ) an algebra in \mathcal{C} and $T : A \otimes A \rightarrow A \otimes A$ a weak pseudotwistor. In view of Example 2.11, we may think of T as some sort of 2-cocycle for A . We will see that we can define as well some sort of 2-coboundaries.

Proposition 2.19 *Let (\mathcal{C}, \otimes) be a strict monoidal category and (A, μ) an algebra in \mathcal{C} . Assume that we are given a triple (f, F, \mathcal{F}) , where $f : A \rightarrow A$, $F : A \otimes A \rightarrow A \otimes A$ and $\mathcal{F} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$ are morphisms in \mathcal{C} satisfying the following conditions:*

$$F \circ (id_A \otimes \mu) = (id_A \otimes \mu) \circ \mathcal{F}, \quad (2.21)$$

$$F \circ (\mu \otimes id_A) = (\mu \otimes id_A) \circ \mathcal{F}, \quad (2.22)$$

$$f \circ \mu \circ F = \mu. \quad (2.23)$$

Then the morphism $D : A \otimes A \rightarrow A \otimes A$, $D = F \circ (f \otimes f)$ is a weak pseudotwistor with weak companion $\mathcal{D} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, $\mathcal{D} = \mathcal{F} \circ (f \otimes f \otimes f)$. We denote $\partial(f, F) = D$ and call such a weak pseudotwistor 2-coboundary for A . Moreover, f is an algebra homomorphism from $A^{\partial(f, F)}$ to A , so in particular if f is invertible then $A^{\partial(f, F)}$ and A are isomorphic as algebras.

Proof. We check (2.1) for D and \mathcal{D} , while (2.2) is similar and left to the reader:

$$\begin{aligned} D \circ (id_A \otimes \mu) \circ (id_A \otimes D) &= F \circ (f \otimes f) \circ (id_A \otimes \mu) \circ (id_A \otimes F) \circ (id_A \otimes f \otimes f) \\ &= F \circ (f \otimes id_A) \circ (id_A \otimes f) \circ (id_A \otimes \mu) \circ (id_A \otimes F) \\ &\quad \circ (id_A \otimes f \otimes f) \\ &= F \circ (f \otimes id_A) \circ (id_A \otimes f \circ \mu \circ F) \circ (id_A \otimes f \otimes f) \\ (2.23) \quad &\stackrel{=}{=} F \circ (f \otimes id_A) \circ (id_A \otimes \mu) \circ (id_A \otimes f \otimes f) \\ &= F \circ (id_A \otimes \mu) \circ (f \otimes id_A \otimes id_A) \circ (id_A \otimes f \otimes f) \\ (2.21) \quad &\stackrel{=}{=} (id_A \otimes \mu) \circ \mathcal{F} \circ (f \otimes f \otimes f) = (id_A \otimes \mu) \circ \mathcal{D}. \end{aligned}$$

The fact that f is an algebra homomorphism $A^{\partial(f, F)} \rightarrow A$ follows immediately from (2.23). \square

Example 2.20 *Let (\mathcal{C}, \otimes) be a strict monoidal category and $R_X : X \rightarrow X$ a family of natural isomorphisms in \mathcal{C} . If (A, μ) is an algebra in \mathcal{C} , then $f := R_A$, $F := R_{A \otimes A}^{-1}$ and $\mathcal{F} := R_{A \otimes A \otimes A}^{-1}$ satisfy the hypotheses of Proposition 2.19. Indeed, the naturality of R implies $R_{A \otimes A} \circ (id_A \otimes \mu) = (id_A \otimes \mu) \circ R_{A \otimes A \otimes A}$, which is (2.21), $R_{A \otimes A} \circ (\mu \otimes id_A) = (\mu \otimes id_A) \circ R_{A \otimes A \otimes A}$, which is (2.22), and $R_A \circ \mu = \mu \circ R_{A \otimes A}$, which is (2.23).*

Another example will be given in the next section.

3 Rota-Baxter type operators

We recall (see for instance the recent survey [11] and references therein) the concept of Rota-Baxter operator. Let (A, μ) be an associative algebra over a field k and $\lambda \in k$ a fixed element. A linear map $R : A \rightarrow A$ is called a Rota-Baxter operator of weight λ if it satisfies the relation

$$R(a)R(b) = R(R(a)b + aR(b) + \lambda ab), \quad \forall a, b \in A. \quad (3.1)$$

If this is the case, the new multiplication $*_\lambda$ on A defined by

$$a *_\lambda b = R(a)b + aR(b) + \lambda ab, \quad \forall a, b \in A,$$

and called the *double product*, is associative and R is an algebra map from $(A, *_\lambda)$ to (A, μ) .

Proposition 3.1 *If $R : A \rightarrow A$ is a Rota-Baxter operator of weight λ on an algebra (A, μ) as above, then the linear map*

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = R(a) \otimes b + a \otimes R(b) + \lambda a \otimes b, \quad \forall a, b \in A, \quad (3.2)$$

is a weak pseudotwistor with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$,

$$\begin{aligned} \mathcal{T}(a \otimes b \otimes c) &= R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes R(c) + a \otimes R(b) \otimes R(c) + \lambda R(a) \otimes b \otimes c \\ &\quad + \lambda a \otimes R(b) \otimes c + \lambda a \otimes b \otimes R(c) + \lambda^2 a \otimes b \otimes c, \end{aligned}$$

*and the new associative product $\mu \circ T$ on A coincides with the double product $*_\lambda$.*

Proof. Obviously $\mu \circ T$ coincides with $*_\lambda$, so we only need to prove that T is a weak pseudotwistor. We compute, for $a, b, c \in A$:

$$\begin{aligned} T \circ (id_A \otimes (\mu \circ T))(a \otimes b \otimes c) &= T(a \otimes R(b)c + a \otimes bR(c) + \lambda a \otimes bc) \\ &= R(a) \otimes R(b)c + a \otimes R(R(b)c) + \lambda a \otimes R(b)c \\ &\quad + R(a) \otimes bR(c) + a \otimes R(bR(c)) + \lambda a \otimes bR(c) \\ &\quad + \lambda R(a) \otimes bc + \lambda a \otimes R(bc) + \lambda^2 a \otimes bc \\ &= R(a) \otimes R(b)c + \lambda a \otimes R(b)c + R(a) \otimes bR(c) + \lambda a \otimes bR(c) \\ &\quad + \lambda R(a) \otimes bc + \lambda^2 a \otimes bc + a \otimes R(R(b)c + bR(c) + \lambda bc) \\ &\stackrel{(3.1)}{=} R(a) \otimes R(b)c + \lambda a \otimes R(b)c + R(a) \otimes bR(c) + \lambda a \otimes bR(c) \\ &\quad + \lambda R(a) \otimes bc + \lambda^2 a \otimes bc + a \otimes R(b)R(c) \\ &= (id_A \otimes \mu) \circ \mathcal{T}(a \otimes b \otimes c). \end{aligned}$$

A similar computation shows that: $T \circ ((\mu \circ T) \otimes id_A)(a \otimes b \otimes c) = (\mu \otimes id_A) \circ \mathcal{T}(a \otimes b \otimes c)$. \square

Let A be an associative algebra over a field k and $\beta, \gamma : A \rightarrow A$ two commuting Rota-Baxter operators of weight 0. It was proved in [2] (as a consequence of the fact that, via β and γ , A becomes a so-called *quadri-algebra*) that the new multiplication defined on A by

$$a * b = \gamma\beta(a)b + \beta(a)\gamma(b) + \gamma(a)\beta(b) + a\gamma\beta(b), \quad \forall a, b \in A, \quad (3.3)$$

is associative. We want to obtain this as a consequence of Corollary 2.17. By Proposition 3.1, we can consider the weak pseudotwistors $T, D : A \otimes A \rightarrow A \otimes A$,

$$T(a \otimes b) = \gamma(a) \otimes b + a \otimes \gamma(b), \quad D(a \otimes b) = \beta(a) \otimes b + a \otimes \beta(b),$$

with weak companions \mathcal{T} and respectively \mathcal{D} defined by

$$\begin{aligned}\mathcal{T}(a \otimes b \otimes c) &= \gamma(a) \otimes \gamma(b) \otimes c + \gamma(a) \otimes b \otimes \gamma(c) + a \otimes \gamma(b) \otimes \gamma(c), \\ \mathcal{D}(a \otimes b \otimes c) &= \beta(a) \otimes \beta(b) \otimes c + \beta(a) \otimes b \otimes \beta(c) + a \otimes \beta(b) \otimes \beta(c).\end{aligned}$$

Since γ and β commute, it is obvious that $T \circ D = D \circ T$, so (2.17) is satisfied. An easy computation shows that (2.18) and (2.19) are also satisfied, so $T \circ D$ is a weak pseudotwistor and obviously $\mu \circ T \circ D$ is exactly the multiplication (3.3).

Example 3.2 Let A be an associative algebra over a field k and $R : A \rightarrow A$ a bijective Rota-Baxter operator of weight λ , with inverse R^{-1} . Then the linear maps $f := R$, $F : A \otimes A \rightarrow A \otimes A$, $F(a \otimes b) = a \otimes R^{-1}(b) + R^{-1}(a) \otimes b + \lambda R^{-1}(a) \otimes R^{-1}(b)$, and $\mathcal{F} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, $\mathcal{F}(a \otimes b \otimes c) = R^{-1}(a) \otimes b \otimes c + a \otimes R^{-1}(b) \otimes c + a \otimes b \otimes R^{-1}(c) + \lambda a \otimes R^{-1}(b) \otimes R^{-1}(c) + \lambda R^{-1}(a) \otimes b \otimes R^{-1}(c) + \lambda R^{-1}(a) \otimes R^{-1}(b) \otimes c + \lambda^2 R^{-1}(a) \otimes R^{-1}(b) \otimes R^{-1}(c)$, satisfy the hypotheses of Proposition 2.19, as one can easily check, and the weak pseudotwistor $\partial(f, F)$ (notation as in Proposition 2.19) is the one defined by (3.2).

Remark 3.3 Bijective Rota-Baxter operators exist. For example, if A is an associative algebra over a field k and $\lambda \in k$, $\lambda \neq 0$, then the (bijective) linear map $R : A \rightarrow A$, $R(a) = -\lambda a$, for all $a \in A$, is a Rota-Baxter operator of weight λ , cf. [13].

Definition 3.4 Let A be an associative algebra over a field k and $\alpha, \beta : A \rightarrow A$ two linear maps. A linear map $R : A \rightarrow A$ will be called an (α, β) -Rota-Baxter operator if the following conditions are satisfied, for all $a, b \in A$:

$$\begin{aligned}\alpha(R(a)R(b)) &= \alpha(R(a))\alpha(R(b)), \\ \beta(R(a)R(b)) &= \beta(R(a))\beta(R(b)), \\ R(a)R(b) &= R(\alpha(R(a))b + a\beta(R(b))).\end{aligned}$$

Obviously, an (id_A, id_A) -Rota-Baxter operator is just a Rota-Baxter operator of weight 0. A nontrivial example (which actually inspired this concept) may be found in [10]: A is the algebra of continuous functions on \mathbb{R} with values in some unital algebra B , q is a number with $0 < q < 1$, $\alpha = id_A$, $\beta = M_q$ is the q -dilation operator and $R = I_q$ is the Jackson q -integral. Then formula (20) in [10] says exactly that R is an (α, β) -Rota-Baxter operator.

Proposition 3.5 If R is an (α, β) -Rota-Baxter operator as above, then the linear map

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = \alpha(R(a)) \otimes b + a \otimes \beta(R(b)), \quad \forall a, b \in A,$$

is a weak pseudotwistor with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$,

$$\mathcal{T}(a \otimes b \otimes c) = \alpha(R(a)) \otimes \alpha(R(b)) \otimes c + \alpha(R(a)) \otimes b \otimes \beta(R(c)) + a \otimes \beta(R(b)) \otimes \beta(R(c)).$$

Consequently, the new multiplication defined on A by the formula $a * b = \alpha(R(a))b + a\beta(R(b))$, for all $a, b \in A$, is associative.

Proof. Follows by a direct computation. □

Example 3.6 Let A be an associative algebra over a field k and $R : A \rightarrow A$ a so-called Reynolds operator (see for instance [22]), that is R satisfies the following condition:

$$R(a)R(b) = R(R(a)b + aR(b) - R(a)R(b)), \quad \forall a, b \in A.$$

If one defines a new multiplication on A , by

$$a * b = R(a)b + aR(b) - R(a)R(b), \quad \forall a, b \in A,$$

then (for instance as a consequence of the theory developped in [21]) $*$ is associative.

If we define the linear map

$$T : A \otimes A \rightarrow A \otimes A, \quad T(a \otimes b) = R(a) \otimes b + a \otimes R(b) - R(a) \otimes R(b), \quad \forall a, b \in A,$$

then one can check, by a direct computation, that T is a weak pseudotwistor, with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, $\mathcal{T}(a \otimes b \otimes c) = R(a) \otimes R(b) \otimes c + R(a) \otimes b \otimes R(c) + a \otimes R(b) \otimes R(c) - 2R(a) \otimes R(b) \otimes R(c)$, and the resulting associative multiplication $\mu \circ T$ coincides with $*$.

We recall from [17] that, if (A, μ) is an associative unital algebra with unit 1_A over a field k , a linear map $P : A \rightarrow A$ is called a *TD-operator* if

$$P(a)P(b) = P(P(a)b + aP(b) - aP(1_A)b), \quad \forall a, b \in A. \quad (3.4)$$

If this is the case, the new multiplication defined on A by $a * b = P(a)b + aP(b) - aP(1_A)b$, for all $a, b \in A$, is associative.

Proposition 3.7 If $P : A \rightarrow A$ is a TD-operator, then the linear map $T : A \otimes A \rightarrow A \otimes A$, $T(a \otimes b) = P(a) \otimes b + a \otimes P(b) - aP(1_A) \otimes b$, for all $a, b \in A$, is a weak pseudotwistor with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, $\mathcal{T}(a \otimes b \otimes c) = P(a) \otimes P(b) \otimes c + P(a) \otimes b \otimes P(c) + a \otimes P(b) \otimes P(c) + aP(1_A) \otimes bP(1_A) \otimes c - aP(1_A) \otimes P(b) \otimes c - aP(1_A) \otimes b \otimes P(c) - P(a) \otimes bP(1_A) \otimes c$, and the associative multiplications $*$ and $\mu \circ T$ coincide.

Proof. A straightforward computation, by using also the identity $P(1_A)P(a) = P(a)P(1_A)$, for all $a \in A$, which follows immediately from (3.4). \square

We recall from [18] that, if (A, μ) is an associative algebra over a field k , a right (respectively left) Baxter operator on A is a linear map $P : A \rightarrow A$ (respectively $Q : A \rightarrow A$) such that $P(a)P(b) = P(P(a)b)$ (respectively $Q(a)Q(b) = Q(aQ(b))$), for all $a, b \in A$. If moreover P and Q commute, then, by [18], Theorem 2.10, the new multiplication defined on A by $a * b = P(a)Q(b)$, for all $a, b \in A$, is associative. By a straightforward computation, one proves the following result:

Proposition 3.8 The linear map $T : A \otimes A \rightarrow A \otimes A$, $T = P \otimes Q$, is a weak pseudotwistor, with weak companion $\mathcal{T} : A \otimes A \otimes A \rightarrow A \otimes A \otimes A$, $\mathcal{T} = P \otimes P \circ Q \otimes Q$, and the associative multiplications $*$ and $\mu \circ T$ coincide.

4 An equivalence relation

Remark 4.1 Let (\mathcal{C}, \otimes) be a strict monoidal category and A an algebra in \mathcal{C} . Then $T = id_{A \otimes A}$ is a weak pseudotwistor for A , with weak companion $\mathcal{T} = id_{A \otimes A \otimes A}$, and $A^T = A$.

Proposition 4.2 *Let (\mathcal{C}, \otimes) be a strict monoidal category and (A, μ) an algebra in \mathcal{C} . Let $T : A \otimes A \rightarrow A \otimes A$ be a weak pseudotwistor for A with weak companion \mathcal{T} and $D : A^T \otimes A^T \rightarrow A^T \otimes A^T$ a weak pseudotwistor for A^T with weak companion \mathcal{D} . Then $T \circ D$ is a weak pseudotwistor for A with weak companion $\mathcal{T} \circ \mathcal{D}$, and $(A^T)^D = A^{T \circ D}$.*

Proof. We prove (2.1), while (2.2) is similar and left to the reader:

$$\begin{aligned} T \circ D \circ (id_A \otimes (\mu \circ T \circ D)) &= T \circ (D \circ (id_A \otimes ((\mu \circ T) \circ D))) \\ &\stackrel{(2.1)}{=} T \circ (id_A \otimes (\mu \circ T)) \circ \mathcal{D} \\ &\stackrel{(2.1)}{=} (id_A \otimes \mu) \circ \mathcal{T} \circ \mathcal{D}. \end{aligned}$$

The fact that $(A^T)^D = A^{T \circ D}$ is obvious. \square

Proposition 4.3 *Let (\mathcal{C}, \otimes) be a strict monoidal category, (A, μ) an algebra in \mathcal{C} and $T : A \otimes A \rightarrow A \otimes A$ a weak pseudotwistor for A with weak companion \mathcal{T} , such that T and \mathcal{T} are invertible. Then T^{-1} is a weak pseudotwistor for A^T with weak companion \mathcal{T}^{-1} , and $(A^T)^{T^{-1}} = A$.*

Proof. We prove (2.1) and leave (2.2) to the reader. We need to prove that

$$T^{-1} \circ (id_A \otimes ((\mu \circ T) \circ T^{-1})) = (id_A \otimes (\mu \circ T)) \circ \mathcal{T}^{-1}.$$

This is obviously equivalent to

$$T \circ (id_A \otimes (\mu \circ T)) = (id_A \otimes \mu) \circ \mathcal{T}.$$

The fact that $(A^T)^{T^{-1}} = A$ is obvious. \square

Proposition 4.4 *Let (\mathcal{C}, \otimes) be a strict monoidal category, (A, μ_A) and (B, μ_B) two algebras in \mathcal{C} , $f : A \rightarrow B$ an algebra isomorphism and $T : A \otimes A \rightarrow A \otimes A$ a weak pseudotwistor for A with weak companion \mathcal{T} . Then $D := (f \otimes f) \circ T \circ (f^{-1} \otimes f^{-1})$ is a weak pseudotwistor for B with weak companion $\mathcal{D} := (f \otimes f \otimes f) \circ \mathcal{T} \circ (f^{-1} \otimes f^{-1} \otimes f^{-1})$, and f is also an algebra isomorphism from A^T to B^D .*

Proof. We prove (2.1) for D and leave (2.2) to the reader:

$$\begin{aligned} D \circ (id_B \otimes (\mu_B \circ D)) &= D \circ (id_B \otimes (\mu_B \circ (f \otimes f) \circ T \circ (f^{-1} \otimes f^{-1}))) \\ &= D \circ (id_B \otimes (f \circ \mu_A \circ T \circ (f^{-1} \otimes f^{-1}))) \\ &= (f \otimes f) \circ T \circ (f^{-1} \otimes f^{-1}) \circ (id_B \otimes (f \circ \mu_A \circ T \circ (f^{-1} \otimes f^{-1}))) \\ &= (f \otimes f) \circ T \circ ((f^{-1} \circ id_B) \otimes (f^{-1} \circ f \circ \mu_A \circ T \circ (f^{-1} \otimes f^{-1}))) \\ &= (f \otimes f) \circ T \circ (f^{-1} \otimes (\mu_A \circ T \circ (f^{-1} \otimes f^{-1}))) \\ &= (f \otimes f) \circ T \circ ((id_A \circ f^{-1}) \otimes (\mu_A \circ T \circ (f^{-1} \otimes f^{-1}))) \\ &= (f \otimes f) \circ T \circ (id_A \otimes (\mu_A \circ T)) \circ (f^{-1} \otimes f^{-1} \otimes f^{-1}) \\ &\stackrel{(2.1)}{=} (f \otimes f) \circ (id_A \otimes \mu_A) \circ \mathcal{T} \circ (f^{-1} \otimes f^{-1} \otimes f^{-1}) \\ &= ((f \circ id_A) \otimes (f \circ \mu_A)) \circ \mathcal{T} \circ (f^{-1} \otimes f^{-1} \otimes f^{-1}) \\ &= (f \otimes (\mu_B \circ (f \otimes f))) \circ \mathcal{T} \circ (f^{-1} \otimes f^{-1} \otimes f^{-1}) \\ &= ((id_B \circ f) \otimes (\mu_B \circ (f \otimes f))) \circ \mathcal{T} \circ (f^{-1} \otimes f^{-1} \otimes f^{-1}) \end{aligned}$$

$$\begin{aligned}
&= (id_B \otimes \mu_B) \circ (f \otimes f \otimes f) \circ \mathcal{T} \circ (f^{-1} \otimes f^{-1} \otimes f^{-1}) \\
&= (id_B \otimes \mu_B) \circ \mathcal{D}, \quad q.e.d.
\end{aligned}$$

The fact that f is an algebra morphism from A^T to B^D follows from the fact that $f \circ \mu_A \circ T = \mu_B \circ (f \otimes f) \circ T = \mu_B \circ D \circ (f \otimes f)$. \square

Definition 4.5 Let (\mathcal{C}, \otimes) be a strict monoidal category and $(A, \mu_A), (B, \mu_B)$ two algebras in \mathcal{C} . We will say that A and B are twist equivalent, and write $A \equiv_t B$, if there exists an invertible weak pseudotwistor T for A , with invertible weak companion \mathcal{T} , such that A^T and B are isomorphic as algebras.

Remark 4.6 In view of Remark 2.4, we have $A \equiv_t B$ if and only if there exists an invertible pseudotwistor T for A , with invertible companions \tilde{T}_1 and \tilde{T}_2 , such that A^T and B are isomorphic as algebras, if and only if there exists an invertible R -matrix T for A , with invertible companions \bar{T}_1 and \bar{T}_2 , such that A^T and B are isomorphic as algebras.

Obviously, two isomorphic algebras are twist equivalent.
As a consequence of the above results, we obtain:

Proposition 4.7 \equiv_t is an equivalence relation.

Example 4.8 In the setting of Example 2.11, if σ (respectively τ) is a convolution invertible left (respectively right) 2-cocycle, then ${}_{\sigma}H \equiv_t H$ and $H_{\tau} \equiv_t H$.

Example 4.9 Let (A, μ_A) and (B, μ_B) be two associative algebras over a field k and $R : B \otimes A \rightarrow A \otimes B$ a twisting map, with Sweedler-type notation $R(b \otimes a) = a_R \otimes b_R$, for $a \in A, b \in B$. We can consider the twisted tensor product $A \otimes_R B$ (cf. [6], [23]), which is the associative algebra structure on the linear space $A \otimes B$ given by the multiplication $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$, for $a, a' \in A, b, b' \in B$. Define the linear map

$$\begin{aligned}
T : (A \otimes B) \otimes (A \otimes B) &\rightarrow (A \otimes B) \otimes (A \otimes B), \\
T((a \otimes b) \otimes (a' \otimes b')) &= (a \otimes b_R) \otimes (a'_R \otimes b').
\end{aligned}$$

By [19], T is a so-called twistor for the associative algebra $A \otimes B$, in particular it is a weak pseudotwistor with weak companion

$$\begin{aligned}
\mathcal{T} : (A \otimes B) \otimes (A \otimes B) \otimes (A \otimes B) &\rightarrow (A \otimes B) \otimes (A \otimes B) \otimes (A \otimes B), \\
\mathcal{T}((a \otimes b) \otimes (a' \otimes b') \otimes (a'' \otimes b'')) &= (a \otimes (b_R)_{\mathcal{R}}) \otimes (a'_R \otimes b'_r) \otimes ((a''_r)_{\mathcal{R}} \otimes b''),
\end{aligned}$$

where r and \mathcal{R} are two more copies of R ; moreover, we have that $A \otimes_R B = (A \otimes B)^T$.

Assume now that R is a bijective map. Then obviously T and \mathcal{T} are also bijective, hence $A \otimes_R B \equiv_t A \otimes B$.

Note added. We used the term "Rota-Baxter type operator" in an informal way, to designate an operator that is "similar" to a Rota-Baxter operator. Professor Li Guo kindly draw out attention to the paper [12], where this term was introduced as a rigorous concept and moreover a conjectural list of possible Rota-Baxter type operators was proposed.

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